JOURNAL OF COMPUTATIONAL PHYSICS 77, 361–383 (1988)

An Approximate Linearised Riemann Solver for the Three-Dimensional Euler Equations for Real Gases Using Operator Splitting*

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Received December 12, 1986; revised September 4, 1987

An approximate (linearised) Riemann solver is presented for the solution of the Euler equations of gas dynamics in three dimensions with a general equation of state. The scheme incorporates operator splitting and is applied to the problem of Mach 3 flow past a forward facing step for some specimen equations of state. © 1988 Academic Press, Inc.

1. INTRODUCTION

Prompted by the work of Roe and Pike [1] a linearised approximate Riemann solver has been proposed by Glaister [2] for the solution of the one-dimensional Euler equations of gas dynamics for a general equation of state. We seek here to extend this scheme to the solution of the three-dimensional Euler equations incorporating the technique of operator splitting, again with a general equation of state. At each stage we shall, as in [2], draw a parallel with the scheme developed by Roe [3] for the ideal gas equation of state. Roe's scheme has proved to be successful in its application to two-dimensional test problems (see Section 4); in particular the problem of Mach 3 flow in a wind tunnel containing a step (see [4]).

In Section 2 we consider the Jacobian of one of the flux functions for the Euler equations with a general equation of state, and in Section 3 we derive an approximate Riemann solver for the solution of these equations. Finally, in Section 4 we describe a two-dimensional test problem and display the numerical results achieved using the scheme of Section 3.

2. EULER EQUATIONS AND THE EQUATION OF STATE

In this section we state the equations of motion for an inviscid compressible fluid in three dimensions for a general equation of state and give the eigenvalues and eigenvectors of the Jacobian of one of the corresponding flux functions.

*This work forms part of the research programme of the Institute for Computational Fluid Dynamics at the Universities of Oxford and Reading and was funded by A.W.R.E., Aldermaston under Contract NSN/13B/2A88719.

0021-9991/88 $3.00

581/77/2-6

2.1. *Equations*

The Euler equations governing the flow of an inviscid, compressible fluid in three dimensions can be written in conservation form as

$$\mathbf{w}_t + \mathbf{F}_x + \mathbf{G}_y + \mathbf{H}_z = \mathbf{0}, \tag{2.1}$$

where

$$\mathbf{w} = (\rho, \rho u, \rho v, \rho w, e)^{\mathrm{T}} \tag{2.2a}$$

$$\mathbf{F}(\mathbf{w}) = (\rho u, p + \rho u^2, \rho uv, \rho uw, u(e+p))^{\mathrm{T}} \tag{2.2b}$$

$$\mathbf{G}(\mathbf{w}) = (\rho v, \rho uv, p + \rho v^2, \rho vw, v(e+p))^{\mathrm{T}} \tag{2.2c}$$

$$\mathbf{H}(\mathbf{w}) = (\rho w, \rho uw, \rho vw, p + \rho w^2, w(e+p))^{\mathrm{T}} \tag{2.2d}$$

and

$$e = \rho i + \tfrac{1}{2}\rho(u^2 + v^2 + w^2). \tag{2.2e}$$

The quantities $\rho = \rho(\mathbf{x}, t)$, $u = u(\mathbf{x}, t)$, $v = v(\mathbf{x}, t)$, $w = w(\mathbf{x}, t)$, $p = p(\mathbf{x}, t)$, $i = i(\mathbf{x}, t)$, and $e = e(\mathbf{x}, t)$ represent the density, velocity in the three coordinate directions, pressure, specific internal energy, and the total energy, respectively, at a general position $\mathbf{x} = (x, y, z)$ in a Cartesian coordinate system and at time t.

Equations (2.1)–(2.2e) represent conservation of mass, momentum, and energy. In addition, we assume that there is an equation of state, specific to each fluid, which can be written in the form

$$p = p(\rho, i), \tag{2.3}$$

and that the first derivatives $\partial p/\partial \rho|_i$ and $\partial p/\partial i|_\rho$ are available. In the case of an ideal gas equation (2.3) takes the form

$$p = (\gamma - 1)\rho i, \tag{2.4}$$

where γ is the ratio of specific heat capacities of the fluid; this is sometimes called a γ-law gas.

We are interested in the solution of the system of hyperbolic equations given by Eqs. (2.1)–(2.3).

2.2. *Jacobian*

We now give the Jacobian A, of the flux function $\mathbf{F}(\mathbf{w})$, given by

$$A = \partial \mathbf{F}/\partial \mathbf{w}, \tag{2.5}$$

and state its eigenvalues and (right) eigenvectors, since this information, together with similar expressions for the Jacobians of $\mathbf{G}$ and $\mathbf{H}$, will form the basis for our approximate Riemann solver.

Defining the enthalpy H by

$$H=\frac{e+p}{\rho}=\frac{p}{\rho}+i+\frac{1}{2}q^2, \tag{2.6}$$

where the fluid speed q is given by

$$q^2=u^2+v^2+w^2, \tag{2.7}$$

Eqs. (?.2a)–(2.2b) lead to the expression for the Jacobian,

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ a^2-u^2-\frac{p_i}{\rho}(H-q^2) & 2u-\frac{up_i}{\rho} & -\frac{vp_i}{\rho} & -\frac{wp_i}{\rho} & \frac{p_i}{\rho} \\ -uv & v & u & 0 & 0 \\ -uw & w & 0 & u & 0 \\ -uH+ua^2-\frac{up_i}{\rho}(H-q^2) & H-\frac{u^2p_i}{\rho} & -\frac{uvp_i}{\rho} & -\frac{uwp_i}{\rho} & u+\frac{up_i}{\rho} \end{bmatrix}, \tag{2.8}$$

where the "sound speed" a is given by

$$a^2=\frac{pp_i}{\rho^2}+p_\rho \tag{2.9}$$

and the shorthand notation $p_\rho=(\partial p/\partial\rho)(\rho,i)|_i$, $p_i=(\partial p/\partial i)(\rho,i)|_\rho$ has been used.

2.3. *Eigenvalues aand Eigenvectors*

The eigenvalues λ_i and corresponding eigenvectors $\mathbf{e}_i$ of A are then found to be

$$\lambda_1=u+a, \qquad \mathbf{e}_1=\begin{pmatrix} 1 \\ u+a \\ v \\ w \\ H+ua \end{pmatrix}=\begin{pmatrix} 1 \\ u+a \\ v \\ w \\ \frac{p}{\rho}+i+\frac{1}{2}u^2+\frac{1}{2}v^2+\frac{1}{2}w^2+ua \end{pmatrix}, \tag{2.10a}$$

$$\lambda_2=u-a, \qquad \mathbf{e}_2=\begin{pmatrix} 1 \\ u-a \\ v \\ w \\ H-ua \end{pmatrix}=\begin{pmatrix} 1 \\ u-a \\ v \\ w \\ \frac{p}{\rho}+i+\frac{1}{2}u^2+\frac{1}{2}v^2+\frac{1}{2}w^2-ua \end{pmatrix}, \tag{2.10b}$$

$$\lambda_3 = u, \qquad \mathbf{e}_3 = \begin{pmatrix} 1 \\ u \\ v \\ w \\ H - \dfrac{\rho a^2}{p_i} \end{pmatrix} = \begin{pmatrix} 1 \\ u \\ v \\ w \\ i + \dfrac{1}{2} u^2 + \dfrac{1}{2} v^2 + \dfrac{1}{2} w^2 - \dfrac{\rho p_\rho}{p_i} \end{pmatrix}, \tag{2.10c}$$

$$\lambda_4 = u, \qquad \mathbf{e}_4 = \begin{pmatrix} 0 \\ 0 \\ v \\ 0 \\ v^2 \end{pmatrix} \tag{2.10d}$$

$$\lambda_5 = u, \qquad \mathbf{e}_5 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ w \\ w^2 \end{pmatrix} \tag{2.10e}$$

We note that in the case of an ideal gas the equation of state (2.3) becomes

$$p = (\gamma - 1)\, \rho i \tag{2.11}$$

giving

$$p_i = (\gamma - 1)\, \rho, \qquad p_\rho = (\gamma - 1)\, i \tag{2.12}$$

and thus

$$\frac{a^2}{\gamma - 1} = \frac{p}{\rho} + i = H - \frac{1}{2} u^2 - \frac{1}{2} v^2 - \frac{1}{2} w^2 = H - \frac{1}{2} q^2. \tag{2.13}$$

In particular, the eigenvector $\mathbf{e}_3$ becomes

$$\mathbf{e}_3 = \begin{pmatrix} 1 \\ u \\ v \\ w \\ \dfrac{1}{2} u^2 + \dfrac{1}{2} v^2 + \dfrac{1}{2} w^2 \end{pmatrix} \tag{2.14}$$

Similar expressions can be found for the Jacobians $\partial \mathbf{G}/\partial \mathbf{w}$, $\partial \mathbf{H}/\partial \mathbf{w}$.

In the next section we develop an approximate Riemann solver using the results of this section.

3. An Approximate Riemann Solver

In this section we develop an approximate Riemann solver for the Euler equations in three dimensions with a general equation of state incorporating the technique of operator splitting. We follow a similar line of reasoning to that of Glaister [2].

We seek to solve Eqs. (2.1)–(2.3) approximately using operator splitting, i.e. we solve successively

$$\mathbf{w}_t + \mathbf{F}_x = \mathbf{0} \tag{3.1a}$$

$$\mathbf{w}_t + \mathbf{G}_y = \mathbf{0} \tag{3.1b}$$

$$\mathbf{w}_t + \mathbf{H}_z = \mathbf{0} \tag{3.1c}$$

along x, y and z coordinate lines, respectively. We consider approximate solutions of Eq. (3.1a); then a similar analysis will give approximate solutions of Eqs. (3.1b)–(3.1c).

3.1. *Wavespeeds for Nearby States*

Consider two adjacent states $\mathbf{w}_L$, $\mathbf{w}_R$ (left and right) close to an average state $\mathbf{w}$, at points L and R on an x coordinate line. We seek coefficients $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$, such that

$$\Delta \mathbf{w} = \sum_{j=1}^{5} \alpha_j \mathbf{e}_j \tag{3.2}$$

to within $\mathcal{O}(\Delta^2)$, where $\Delta(\cdot) = (\cdot)_R - (\cdot)_L$.

Following some lengthy algebra, and using the assumption that the left and right states $\mathbf{w}_L$, $\mathbf{w}_R$ are close to the average state $\mathbf{w}$, so that, to within $\mathcal{O}(\Delta^2)$,

$$\Delta(\rho U) = U\Delta\rho + \rho\Delta U, \qquad U = u, v, w, \quad \text{or} \quad i \qquad \text{(3.3a)–(3.3d)}$$

$$\Delta(\rho U^2) = U^2\Delta\rho + 2\rho U\Delta U, \qquad U = u, v, \quad \text{or} \quad w \qquad \text{(3.4a)–(3.4c)}$$

and

$$\Delta p = p_\rho \Delta\rho + p_i \Delta i, \tag{3.5}$$

Eqs. (3.2) give the following expressions for $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, and α_5,

$$\alpha_1 = \frac{1}{2a^2}(\Delta p + \rho a \Delta u) \tag{3.6a}$$

$$\alpha_2 = \frac{1}{2a^2}(\Delta p - \rho a \Delta u) \tag{3.6b}$$

$$\alpha_3 = \Delta \rho - \frac{\Delta p}{a^2} \tag{3.6c}$$

$$\alpha_4 = \frac{\rho}{v} \Delta v \tag{3.6d}$$

$$\alpha_5 = \frac{\rho}{w} \Delta w. \tag{3.6e}$$

We have found $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ such that

$$\Delta \mathbf{w} = \sum_{j=1}^{5} \alpha_j \mathbf{e}_j \tag{3.7}$$

to within $\mathcal{O}(\Delta^2)$, and a routine calculation verifies that

$$\Delta \mathbf{F} = \sum_{j=1}^{5} \lambda_j \alpha_j \mathbf{e}_j \tag{3.8}$$

to within $\mathcal{O}(\Delta^2)$. We are now in a position to construct the approximate Riemann solver.

3.2. *Decomposition for General* $\mathbf{w}_L$, $\mathbf{w}_R$

Consider the algebraic problem of finding average eigenvalues $\tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3, \tilde{\lambda}_4, \tilde{\lambda}_5$ and corresponding average eigenvectors $\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3, \tilde{\mathbf{e}}_4, \tilde{\mathbf{e}}_5$ such that relations (3.7) and (3.8) hold exactly for arbitrary states $\mathbf{w}_L, \mathbf{w}_R$, not necessarily close. Specifically, we seek averages $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}_i, \tilde{p}_\rho, \tilde{p}$, and $\tilde{\imath}$ in terms of two adjacent states $\mathbf{w}_L, \mathbf{w}_R$ (on an x-coordinate line) such that

$$\Delta \mathbf{w} = \sum_{j=1}^{5} \tilde{\alpha}_j \tilde{\mathbf{e}}_j \tag{3.9}$$

and

$$\Delta \mathbf{F} = \sum_{j=1}^{5} \tilde{\lambda}_j \tilde{\alpha}_j \tilde{\mathbf{e}}_j, \tag{3.10}$$

where

$$\Delta(\cdot) = (\cdot)_R - (\cdot)_L \tag{3.11a}$$

$$\mathbf{w} = (\rho, \rho u, \rho v, \rho w, e)^{\mathrm{T}} \tag{3.11b}$$

$$\mathbf{F}(\mathbf{w}) = (\rho u, p + \rho u^2, \rho uv, \rho uw, u(e+p))^{\mathrm{T}} \tag{3.11c}$$

$$e=\rho i+\frac{1}{2}\rho u^2+\frac{1}{2}\rho v^2+\frac{1}{2}\rho w^2 \tag{3.11d}$$

$$p=p(\rho, i) \tag{3.11e}$$

$$\tilde{\lambda}_{1,2,3,4,5}=\tilde{u}+\tilde{a}, \tilde{u}-\tilde{a}, \tilde{u}, \tilde{u}, \tilde{u} \tag{3.12a}$$

$$\tilde{\mathbf{e}}_{1,2}=\begin{pmatrix} 1 \\ \tilde{u}\pm\tilde{a} \\ \tilde{v} \\ \tilde{w} \\ \dfrac{\tilde{p}}{\tilde{\rho}}+\tilde{\imath}+\dfrac{1}{2}\tilde{u}^2+\dfrac{1}{2}\tilde{v}^2+\dfrac{1}{2}\tilde{w}^2\pm\tilde{u}\tilde{a} \end{pmatrix} \tag{3.12b}$$

$$\tilde{\mathbf{e}}_{3}=\begin{pmatrix} 1 \\ \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{\imath}+\dfrac{1}{2}\tilde{q}^2-\dfrac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i} \end{pmatrix} \tag{3.12c}$$

$$\tilde{\mathbf{e}}_{4,5}=\begin{pmatrix} 0 \\ 0 \\ \tilde{v} \\ 0 \\ \tilde{v}^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ \tilde{w} \\ \tilde{w}^2 \end{pmatrix} \tag{3.12d}$$

$$\tilde{\alpha}_1=\frac{1}{2\tilde{a}^2}(\Delta p+\tilde{\rho}\tilde{a}\Delta u) \tag{3.13a}$$

$$\tilde{\alpha}_2=\frac{1}{2\tilde{a}^2}(\Delta p-\tilde{\rho}\tilde{a}\Delta u) \tag{3.13b}$$

$$\tilde{\alpha}_3=\Delta\rho-\frac{\Delta p}{\tilde{a}^2} \tag{3.13c}$$

$$\tilde{\alpha}_4=\frac{\tilde{\rho}}{\tilde{v}}\Delta v \tag{3.13d}$$

$$\tilde{\alpha}_5=\frac{\tilde{\rho}}{\tilde{w}}\Delta w \tag{3.13e}$$

and $\tilde{a}$ is given by

$$\tilde{\rho}\tilde{a}^2 = \frac{\tilde{p}\tilde{p}_i}{\tilde{\rho}} + \tilde{\rho}\tilde{p}_\rho . \tag{3.14}$$

The problem of finding averages $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $\tilde{p}_i, \tilde{p}_\rho, \tilde{p}$, and $\tilde{\imath}$ subject to Eqs. (3.9)–(3.14) will subsequently be denoted by (*). (N.B. The quantities $\tilde{p}_i$ and $\tilde{p}_\rho$ denote approximations to the partial derivatives p_i and p_ρ, respectively.)

The solution of problem (*) will be sought in a similar way to that given by Glaister [2] in one dimension and by Roe and Pike [1] in the specialised, ideal gas case. We note that problem (*) is equivalent to seeking an approximation to the Jacobian A, namely $\tilde{A}$ with eigenvalues $\tilde{\lambda}_i$ and eigenvectors $\tilde{\mathbf{e}}_i$, such that

$$\Delta \mathbf{F} = \tilde{A} \Delta \mathbf{w}$$

which is an alternative approach used in the ideal gas case by Roe [3].

The first step in the analysis of problem (*) is to write out Eqs. (3.9) and (3.10) explicitly, namely,

$$\Delta\rho = \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 \tag{3.15a}$$

$$\Delta(\rho u) = \tilde{\alpha}_1(\tilde{u}+\tilde{a}) + \tilde{\alpha}_2(\tilde{u}-\tilde{a}) + \tilde{\alpha}_3\tilde{u} \tag{3.15b}$$

$$\Delta(\rho v) = \tilde{\alpha}_1\tilde{v} + \tilde{\alpha}_2\tilde{v} + \tilde{\alpha}_3\tilde{v} + \tilde{\alpha}_4\tilde{v} \tag{3.15c}$$

$$\Delta(\rho w) = \tilde{\alpha}_1\tilde{w} + \tilde{\alpha}_2\tilde{w} + \tilde{\alpha}_3\tilde{w} + \tilde{\alpha}_5\tilde{w} \tag{3.15d}$$

$$\begin{aligned}\Delta e = \Delta(\rho i) + \Delta\left(\frac{\rho q^2}{2}\right) &= \tilde{\alpha}_1\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{\imath} + \frac{1}{2}\tilde{q}^2 + \tilde{u}\tilde{a}\right) \\ &+ \tilde{\alpha}_2\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{\imath} + \frac{1}{2}\tilde{q}^2 - \tilde{u}\tilde{a}\right) \\ &+ \tilde{\alpha}_3\left(\tilde{\imath} + \frac{1}{2}\tilde{q}^2 - \frac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i}\right) \\ &+ \tilde{\alpha}_4\tilde{v}^2 + \tilde{\alpha}_5\tilde{w}^2\end{aligned} \tag{3.15e}$$

$$\Delta(\rho u) = \tilde{\alpha}_1(\tilde{u}+\tilde{a}) + \tilde{\alpha}_2(\tilde{u}-\tilde{a}) + \tilde{\alpha}_3\tilde{u} \tag{3.15f}$$

$$\Delta(p+\rho u^2) = \Delta p + \Delta(\rho u^2) = \tilde{\alpha}_1(\tilde{u}+\tilde{a})^2 + \tilde{\alpha}_2(\tilde{u}-\tilde{a})^2 + \tilde{\alpha}_3\tilde{u}^2 \tag{3.15g}$$

$$\Delta(\rho uv) = \tilde{\alpha}_1(\tilde{u}+\tilde{a})\,\tilde{v} + \alpha_2(\tilde{u}-\tilde{a})\,\tilde{v} + \tilde{\alpha}_3\tilde{u}\tilde{v} + \tilde{\alpha}_4\tilde{u}\tilde{v} \tag{3.15h}$$

$$\Delta(\rho uw) = \tilde{\alpha}_1(\tilde{u}+\tilde{a})\,\tilde{w} + \tilde{\alpha}_2(\tilde{u}-\tilde{a})\,\tilde{w} + \tilde{\alpha}_3\tilde{u}\tilde{w} + \tilde{\alpha}_5\tilde{u}\tilde{w} \tag{3.15i}$$

$$\begin{aligned}\Delta(u(e+p)) &= \Delta(\rho u i) + \Delta\left(\frac{\rho u q^2}{2}\right) + \Delta(up)\\ &= \tilde{\alpha}_1(\tilde{u}+\tilde{a})\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{\imath} + \frac{1}{2}\tilde{q}^2 + \tilde{u}\tilde{a}\right)\\ &\quad + \tilde{\alpha}_2(\tilde{u}-\tilde{a})\left(\frac{\tilde{p}}{\tilde{\rho}} + \tilde{\imath} + \frac{1}{2}\tilde{q}^2 - \tilde{u}\tilde{a}\right)\\ &\quad + \tilde{\alpha}_3\,\tilde{u}\left(\tilde{\imath} + \frac{1}{2}\tilde{q}^2 - \frac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i}\right)\\ &\quad + \tilde{\alpha}_4\tilde{u}\tilde{v}^2 + \tilde{\alpha}_5\tilde{u}\tilde{w}^2, \end{aligned} \tag{3.15j}$$

where

$$q^2 = u^2 + v^2 + w^2 \tag{3.16}$$

as before, and for convenience we have written

$$\tilde{q}^2 = \tilde{u}^2 + \tilde{v}^2 + \tilde{w}^2. \tag{3.17}$$

Equation (3.15a) is satisfied by any average we care to define, while Eq. (3.15f) is the same as Eq. (3.15b); thus it remains to solve equations (3.15c)–(3.15j). From Eq. (3.15f) we have

$$\begin{aligned}\Delta(\rho u) &= \tilde{u}(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) + \tilde{a}(\tilde{\alpha}_1 - \tilde{\alpha}_2)\\ &= \tilde{u}\Delta\rho + \tilde{\rho}\Delta u,\end{aligned} \tag{3.18}$$

and from Eq. (3.15g) we obtain

$$\begin{aligned}\Delta(\rho u^2) &= \tilde{u}^2(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) + 2\tilde{u}\tilde{a}(\tilde{\alpha}_1 - \tilde{\alpha}_2)\\ &= \tilde{u}^2\Delta\rho + 2\tilde{u}\tilde{a}\Delta u.\end{aligned} \tag{3.19}$$

Substituting for $\tilde{\rho}$ from Eq. (3.18) into Eq. (3.19) yields the quadratic equation for $\tilde{u}$,

$$\tilde{u}^2\Delta\rho - 2\tilde{u}\Delta(\rho u) + \Delta(\rho u^2) = 0. \tag{3.20}$$

Only one solution of Eq. (3.20) is productive, namely

$$\begin{aligned}\tilde{u} &= \frac{\Delta(\rho u) - \sqrt{(\Delta(\rho u))^2 - \Delta\rho\Delta(\rho u^2)}}{\Delta\rho}\\ &= \frac{\sqrt{\rho_L}\,u_L + \sqrt{\rho_R}\,u_R}{\sqrt{\rho_L} + \sqrt{\rho_R}}\end{aligned} \tag{3.21}$$

which, on substituting $\tilde{u}$ into Eq. (3.18) gives

$$\tilde{\rho}=\frac{\Delta(\rho u)-\tilde{u}\Delta\rho}{\Delta u}=\sqrt{\rho_L\rho_R}, \tag{3.22}$$

as found in the one-dimensional case in [2].

From Eqs. (3.15c)–(3.15d) we have

$$\Delta(\rho v)=\tilde{v}\Delta\rho+\tilde{\rho}\Delta v \tag{3.23a}$$

$$\Delta(\rho w)=\tilde{w}\Delta\rho+\tilde{\rho}\Delta w; \tag{3.23b}$$

i.e.,

$$\tilde{v}=\frac{\Delta(\rho v)-\tilde{\rho}\Delta v}{\Delta\rho}=\frac{\sqrt{\rho_L}\,v_L+\sqrt{\rho_R}\,v_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}, \tag{3.24a}$$

$$\tilde{w}=\frac{\Delta(\rho w)-\tilde{\rho}\Delta w}{\Delta\rho}=\frac{\sqrt{\rho_L}\,w_L+\sqrt{\rho_R}\,w_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}. \tag{3.24b}$$

We have now determined $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, and $\tilde{w}$, and we can now show that

$$\Delta(\rho U^2)-2\tilde{\rho}\tilde{U}\Delta U-\tilde{U}^2\Delta\rho=0, \qquad U=u,\ v, \quad \text{or} \quad w \qquad \text{(3.25a)–(3.25c)}$$

$$\Delta(\rho uV)-\tilde{\rho}\tilde{u}\Delta V-\tilde{V}\tilde{u}\Delta\rho-\tilde{\rho}\tilde{V}\Delta u=0, \qquad V=v \quad \text{or} \quad w \qquad \text{(3.26a)–(3.26b)}$$

$$\Delta\left(\frac{\rho uU^2}{2}\right)-\frac{\tilde{u}\tilde{U}^2}{2}\Delta\rho-\tilde{\rho}\tilde{u}\tilde{U}\Delta U-\frac{\tilde{\rho}\tilde{U}^2}{2}\Delta u$$

$$=\frac{\tilde{\rho}^2(\Delta U)^2\,\Delta u}{2(\sqrt{\rho_R}+\sqrt{\rho_L})^2}, \qquad U=u,\ v, \quad \text{or} \quad w \qquad \text{(3.27a)–(3.27c)}$$

$$\Delta(up)-\tilde{u}\Delta p=\tilde{\rho}\Delta u\left(\sqrt{\rho_L}\frac{p_L}{\rho_L}+\sqrt{\rho_R}\frac{p_R}{\rho_R}\right)\Big/(\sqrt{\rho_L}+\sqrt{\rho_R}), \tag{3.28}$$

and

$$\frac{\sqrt{\rho_L}\,U_L^2+\sqrt{\rho_R}\,U_R^2}{\sqrt{\rho_L}+\sqrt{\rho_R}}-\tilde{U}^2=\frac{\tilde{\rho}(\Delta U)^2}{(\sqrt{\rho_L}+\sqrt{\rho_R})^2}, \qquad U=u,\ v, \quad \text{or} \quad w \qquad \text{(3.29a)–(3.29c)}$$

all of which will be used later. From Eqs. (3.26a)–(3.26b) and (3.23a)–(3.23e) we can see that Eqs. (3.15h)–(3.15i) are automatically satisfied. We are now left with equations (3.15e) and (3.15j). Before we study the remaining two equations we note two important identities. Using Eqs. (3.16), (3.17), and (3.25a)–(3.25c) we see that

$$\Delta\left(\frac{\rho q^2}{2}\right)-\tilde{\rho}\tilde{u}\Delta u-\tilde{\rho}\tilde{v}\Delta v-\tilde{\rho}\tilde{w}\Delta w-\frac{1}{2}\tilde{q}^2\,\Delta\rho=0, \tag{3.30}$$

and using Eqs. (3.16), (3.17), and (3.27a)–(3.27c) we see that

$$\Delta\left(\frac{\rho u q^2}{2}\right)-\frac{\tilde{u}\tilde{q}^2}{2}\Delta\rho-\tilde{\rho}\tilde{u}(\tilde{u}\Delta u+\tilde{v}\Delta v+\tilde{w}\Delta w)-\frac{\tilde{\rho}\tilde{q}^2}{2}\Delta u$$

$$=\tilde{\rho}^2\,\frac{((\Delta u)^2+(\Delta v)^2+(\Delta w)^2)\,\Delta u}{2(\sqrt{\rho_L}+\sqrt{\rho_R})^2}. \tag{3.31}$$

We begin by rewriting Eqs. (3.15e) and (3.15j), using Eqs. (3.13a)–(3.14) and (3.30), to give

$$\Delta(\rho i)-\tilde{\imath}\Delta\rho-\frac{\tilde{p}\Delta p}{\tilde{\rho}\tilde{a}^2}+\tilde{\alpha}_3\frac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i}=0 \tag{3.32}$$

and

$$\begin{aligned}&\Delta(\rho u i)-\tilde{u}\tilde{\imath}\Delta\rho-\tilde{\rho}\tilde{\imath}\Delta u+\Delta(up)-\tilde{u}\Delta p-\tilde{p}\Delta u\\ &\quad+\Delta\left(\frac{\rho u q^2}{2}\right)-\frac{\tilde{u}\tilde{q}^2}{2}\Delta\rho-\frac{\tilde{\rho}\tilde{q}^2}{2}\Delta u\\ &\quad-\tilde{\rho}\tilde{u}(\tilde{u}\Delta u+\tilde{v}\Delta v+\tilde{w}\Delta w)-\frac{\tilde{u}\tilde{p}}{\tilde{\rho}\tilde{a}^2}\Delta p\\ &\quad+\tilde{\alpha}_3\tilde{u}\frac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i}=0.\end{aligned} \tag{3.33}$$

Now, subtracting Eq. (3.32) multiplied by $\tilde{u}$ from Eqs. (3.33) and using Eqs. (3.28), (3.29a)–(3.29c), and (3.31) together with the identity

$$\Delta(\rho u i)-\tilde{u}\Delta(\rho i)=\tilde{\rho}\Delta u\,\frac{(\sqrt{\rho_L}\,i_L+\sqrt{\rho_R}\,i_R)}{\sqrt{\rho_L}+\sqrt{\rho_R}}, \tag{3.34}$$

we obtain, after division by $\tilde{\rho}\Delta u$,

$$\frac{\tilde{p}}{\tilde{\rho}}+\tilde{\imath}+\frac{1}{2}\tilde{q}^2=\left(\sqrt{\rho_L}\left(\frac{p_L}{\rho_L}+i_L+\frac{1}{2}q_L^2\right)+\sqrt{\rho_R}\left(\frac{p_R}{\rho_R}+i_R+\frac{1}{2}q_R^2\right)\right)\Big/(\sqrt{\rho_L}+\sqrt{\rho_R}), \tag{3.35}$$

where $q^2_{L(R)}=u^2_{L(R)}+v^2_{L(R)}+w^2_{L(R)}$. Therefore, if we define a mean enthalpy $\tilde{H}$, by

$$\tilde{H}=\frac{\tilde{p}}{\tilde{\rho}}+\tilde{\imath}+\frac{1}{2}\tilde{q}^2, \tag{3.36}$$

we find, from Eq. (3.35), that

$$\tilde{H}=\frac{\sqrt{\rho_L}H_L+\sqrt{\rho_R}H_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}. \tag{3.37}$$

We have now specified $\tilde{\rho}$, $\tilde{u}$, $\tilde{v}$, $\tilde{w}$, $(\tilde{q})$, $\tilde{p}/\tilde{\rho}+\tilde{\imath}$, and now, in order to specify $\tilde{p}_i, \tilde{p}_\rho, \tilde{\imath}$ (and hence $\tilde{p}$, $\tilde{a}$), we focus attention on Eq. (3.32) which can be written as

$$\Delta(\rho i) - \tilde{\imath}\Delta\rho - \tilde{\rho}\Delta i + \frac{\tilde{\rho}}{\tilde{p}_i}(\tilde{p}_i \Delta i + \tilde{p}_\rho \Delta\rho - \Delta p) = 0. \tag{3.38}$$

A number of choices can now be made, but it appears that the most natural choice is to take

$$\Delta(\rho i) - \tilde{\imath}\Delta\rho - \tilde{\rho}\Delta i = 0; \tag{3.39}$$

i.e.,

$$\tilde{\imath} = \frac{\Delta(\rho i) - \tilde{\rho}\Delta i}{\Delta\rho} = \frac{\sqrt{\rho_L}\, i_L + \sqrt{\rho_R}\, i_R}{\sqrt{\rho_R} + \sqrt{\rho_L}}, \tag{3.40}$$

in which case (3.38) gives

$$\Delta p = \tilde{p}_i \Delta i + \tilde{p}_\rho \Delta\rho \tag{3.41}$$

as a necessary condition. Finally, all we need to complete our approximate Riemann solver is to choose approximations $\tilde{p}_i, \tilde{p}_\rho$ to p_i, p_ρ such that (3.41) holds. In [2] this is seen to be a straightforward matter and we repeat here the proposed approximations $\tilde{p}_i, \tilde{p}_\rho$:

$$\tilde{p}_i = \begin{cases} \dfrac{1}{\Delta i}\left(\dfrac{1}{2}[p(\rho_R, i_R) + p(\rho_L, i_R)] - \dfrac{1}{2}[p(\rho_R, i_L) + p(\rho_L, i_L)]\right) & \text{if } \Delta i \neq 0 \quad (3.42\text{a}) \\ \dfrac{1}{2}[p_i(\rho_L, i) + p_i(\rho_R, i)] \quad \text{if} \quad \Delta i = 0, \quad i_L = i_R = i & (3.42\text{b}) \end{cases}$$

$$\tilde{p}_\rho = \begin{cases} \dfrac{1}{\Delta\rho}\left(\dfrac{1}{2}[p(\rho_R, i_R) + p(\rho_R, i_L)] - \dfrac{1}{2}[p(\rho_L, i_R) + p(\rho_L, i_L)]\right) & \text{if } \Delta\rho \neq 0 \quad (3.43\text{a}) \\ \dfrac{1}{2}[p_\rho(\rho, i_L) + p_\rho(\rho, i_R)] \quad \text{if} \quad \Delta\rho = 0, \rho_L = \rho_R = \rho. & (3.43\text{b}) \end{cases}$$

(N.B. In practice we would replace the conditions $\Delta\rho = 0$, $\Delta i = 0$ by $|\Delta\rho| \leqslant 10^{-m}$, $|\Delta i| \leqslant 10^{-m}$, where the integer m is machine dependent.) All four combinations arising from Eqs. (3.42a)–(3.43b) satisfy Eq. (3.41).

By symmetry, similar results hold for the Jacobians $\partial\mathbf{G}/\partial\mathbf{w}$, $\partial\mathbf{H}/\partial\mathbf{w}$.

Summarising, we can now apply a three-dimensional Riemann solver for the Euler equations with a general equation of state using the technique of operator splitting. We incorporate the results found here, together with the one-dimensional scalar algorithm given in [1], and perform a sequence of one-dimensional calculations along computational grid lines in the x, y, and z-directions in turn. The algorithm along a line $y = \text{constant}$, $z = \text{constant}$ can be described as follows.

Suppose at time level n we have data $\mathbf{w}_L, \mathbf{w}_R$ given at either end of the cell (x_L, x_R) (on a line $y=y_0, z=z_0$), then we update $\mathbf{w}$ to time level $n+1$ in an upwind manner. Thus we

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\mathbf{e}}_j \quad \text{to} \quad \mathbf{w}_R \quad \text{if} \quad \tilde{\lambda}_j>0$$

or

$$\text{add} \quad -\frac{\Delta t}{\Delta x}\tilde{\lambda}_j\tilde{\alpha}_j\tilde{\mathbf{e}}_j \quad \text{to} \quad \mathbf{w}_L \quad \text{if} \quad \tilde{\lambda}_j<0,$$

where $\Delta x = x_R - x_L$, Δt is the time interval from level n to $n+1$, and $\tilde{\lambda}_j$, $\tilde{\alpha}_j$, $\tilde{\mathbf{e}}_j$ are given by

$$\tilde{\lambda}_{1,2,3,4,5} = \tilde{u}+\tilde{a}, \tilde{u}-\tilde{a}, \tilde{u}, \tilde{u}, \tilde{u}$$

$$\tilde{\mathbf{e}}_{1,2,3,4,5} = \begin{bmatrix} 1 \\ \tilde{u}+\tilde{a} \\ \tilde{v} \\ \tilde{w} \\ \dfrac{\tilde{p}}{\tilde{\rho}}+\tilde{\imath}+\dfrac{1}{2}\tilde{q}^2+\tilde{u}\tilde{a} \end{bmatrix}, \begin{bmatrix} 1 \\ \tilde{u}-\tilde{a} \\ \tilde{v} \\ \tilde{w} \\ \dfrac{\tilde{p}}{\tilde{\rho}}+\tilde{\imath}+\dfrac{1}{2}\tilde{q}^2-\tilde{u}\tilde{a} \end{bmatrix}, \begin{bmatrix} 1 \\ \tilde{u} \\ \tilde{v} \\ \tilde{w} \\ \tilde{\imath}+\dfrac{1}{2}\tilde{q}^2-\dfrac{\tilde{\rho}\tilde{p}_\rho}{\tilde{p}_i} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ \tilde{v} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \tilde{w} \end{bmatrix}$$

$$\tilde{\alpha}_{1,2,3,4,5} = \frac{1}{2\tilde{a}^2}(\Delta p+\tilde{\rho}\tilde{a}\Delta u), \frac{1}{2\tilde{a}^2}(\Delta p-\tilde{\rho}\tilde{a}\Delta u), \Delta\rho-\frac{\Delta p}{\tilde{a}^2}, \tilde{\rho}\Delta v, \tilde{\rho}\Delta w$$

$$\tilde{\rho}=\sqrt{\rho_L\rho_R}, \qquad \tilde{U}=\frac{\sqrt{\rho_L}U_L+\sqrt{\rho_R}U_R}{\sqrt{\rho_L}+\sqrt{\rho_R}}, \qquad U=u, v, w, i, \quad \text{or} \quad H$$

$$\tilde{q}^2=\tilde{u}^2+\tilde{v}^2+\tilde{w}^2, \qquad \tilde{p}=\tilde{\rho}\left(\tilde{H}-\tilde{\imath}-\frac{1}{2}\tilde{q}^2\right), \qquad \tilde{a}^2=\frac{\tilde{p}\tilde{p}_i}{\tilde{\rho}^2}+\tilde{p}_\rho;$$

$\tilde{p}_i, \tilde{p}_\rho$ are given by Eqs. (3.42a)–(3.43b) and $\Delta(\cdot)=(\cdot)_R-(\cdot)_L$. We note that factors $\tilde{v}, \tilde{w}$ have been taken out of $\tilde{\mathbf{e}}_4, \tilde{\mathbf{e}}_5$ so that $\tilde{\alpha}_4, \tilde{\alpha}_5$ will not become indeterminate. Similar results apply for updating in the y and z directions.

The Riemann solver we have constructed in this section is a conservative algorithm (when incorporated with operator splitting) and has the important one-dimensional shock recognising property guaranteed by Eqs. (3.9), (3.10). Problems will occur, as with all operator split schemes, when attempting to capture a shock that is oblique to the grid. Results for a one-dimensional test problem can be found in [2].

In the next section we describe a two-dimensional test problem, and display the numerical results achieved using the scheme of this section.

4. A Test Problem and the Numerical Results

In this section we describe a standard test problem in two-dimensional gas dynamics, and give the numerical results achieved for this problem using the Riemann solver described in Section 3.

The test problem we consider was originally introduced by Emery [4], but has recently been reviewed by Woodward and Colella [5]. The problem begins with uniform Mach 3 flow in a tunnel containing a step. The tunnel is 3 units long and 1 unit wide. The step is 0.2 units high and is located 0.6 units from the left-hand end of the tunnel. At the left an inflow boundary condition is applied, and at the right, where the exit velocity is always supersonic, all gradients are assumed to vanish. We assume slab symmetry, i.e., in the direction orthogonal to the plane of computation the tunnel is assumed to have infinite width.

The equations of motion governing the flow are the two-dimensional Euler equations, namely

$$\mathbf{w}_t + \mathbf{F}_x + \mathbf{G}_y = \mathbf{0}, \tag{4.1a}$$

where

$$\mathbf{w} = (\rho, \rho u, \rho v, e)^{\mathrm{T}} \tag{4.1b}$$

$$\mathbf{F}(\mathbf{w}) = (\rho u, p + \rho u^2, \rho uv, u(e+p))^{\mathrm{T}} \tag{4.1c}$$

$$\mathbf{G}(\mathbf{w}) = (\rho v, \rho uv, p + \rho v^2, v(e+p))^{\mathrm{T}} \tag{4.1d}$$

$$e = \rho i + \tfrac{1}{2}\rho(u^2 + v^2) \tag{4.1e}$$

with

$$p = p(\rho, i), \tag{4.1f}$$

where the particular form for Eq. (4.1f) is given, and the flow variables are all functions of (x, y, t).

The initial conditions for the gas in the tunnel are given by

$$\rho(x, y, 0) = \rho_0 = 1.4$$

$$u(x, y, 0) = u_0 = 3$$

$$v(x, y, 0) = v_0 = 0$$

$$p(x, y, 0) = p_0 = 1$$

all x, y and hence $i(x, y, 0) = i_0$ from the equation of state $p_0 = p(\rho_0, i_0)$. Gas is continually fed in at the left-hand boundary with the flow variables given by $(\rho, u, v, p) = (\rho_0, u_0, v_0, p_0)$ (see Fig. 1).

Along the walls of the tunnel we apply reflecting boundary conditions.

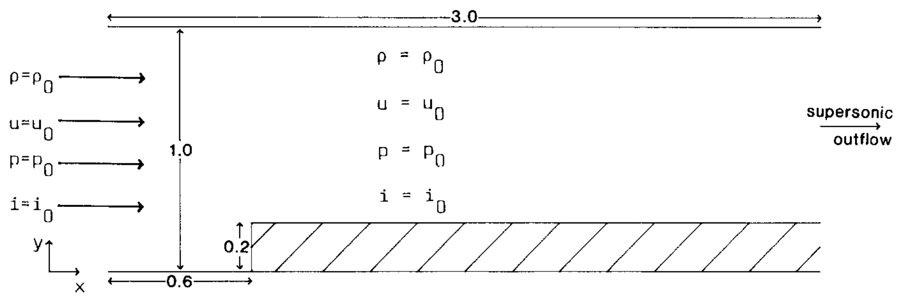

FIG. 1. Geometry of the wind tunnel with a step.

Specifically, along a boundary given by $x = \text{constant}$, we consider an image cell and impose equal density, pressure, and tangential velocity, and equal and opposite normal velocity at either end of the cell, i.e., ρ, p, v, u, respectively, in this case. A similar argument applies for a reflecting boundary given by $y = \text{constant}$.

We consider three equations of state for the gas: (a) the ideal equation of state; (b) the stiffened equation of state; and (c) an equation of state for equilibrium air.

(a) Ideal equation of state. This can be written in the general form

$$p = (\gamma - 1)\,\rho i,$$

where γ is a constant and represents the ratio of specific heat capacities of the fluid.

(b) Stiffened equation of state. This is usually written in the form

$$p = B(\rho/\rho_0 - 1) + (\gamma - 1)\,\rho i,$$

where B is a constant, and ρ_0 represents a reference density.

(c) "Real air" equation of state. One form of the equation of state for equilibrium air is given by Srinivasan, Tannehill, and Weilmuenster [6] and can be written as

$$p = (\tilde{\gamma} - 1)\,\rho i,$$

where

$$\begin{aligned}\tilde{\gamma} = \tilde{\gamma}(\rho, i) = {} & a_1 + a_2 Y + a_3 Z + a_4 YZ + a_5 Y^2 + a_6 Z^2 + a_7 Y^2 Z + a_8 YZ^2 \\ & + a_9 Y^3 + a_{10} Z^3 + (a_{11} + a_{12} Y + a_{13} Z + a_{14} YZ + a_{15} Y^2 \\ & + a_{16} Z^2 + a_{17} Y^2 Z + a_{18} YZ^2 + a_{19} Y^3 + a_{20} Z^3)/ \\ & (1 + {}_e(a_{21} + a_{22} Y + a_{23} Z + a_{23}\, YZ))\end{aligned}$$

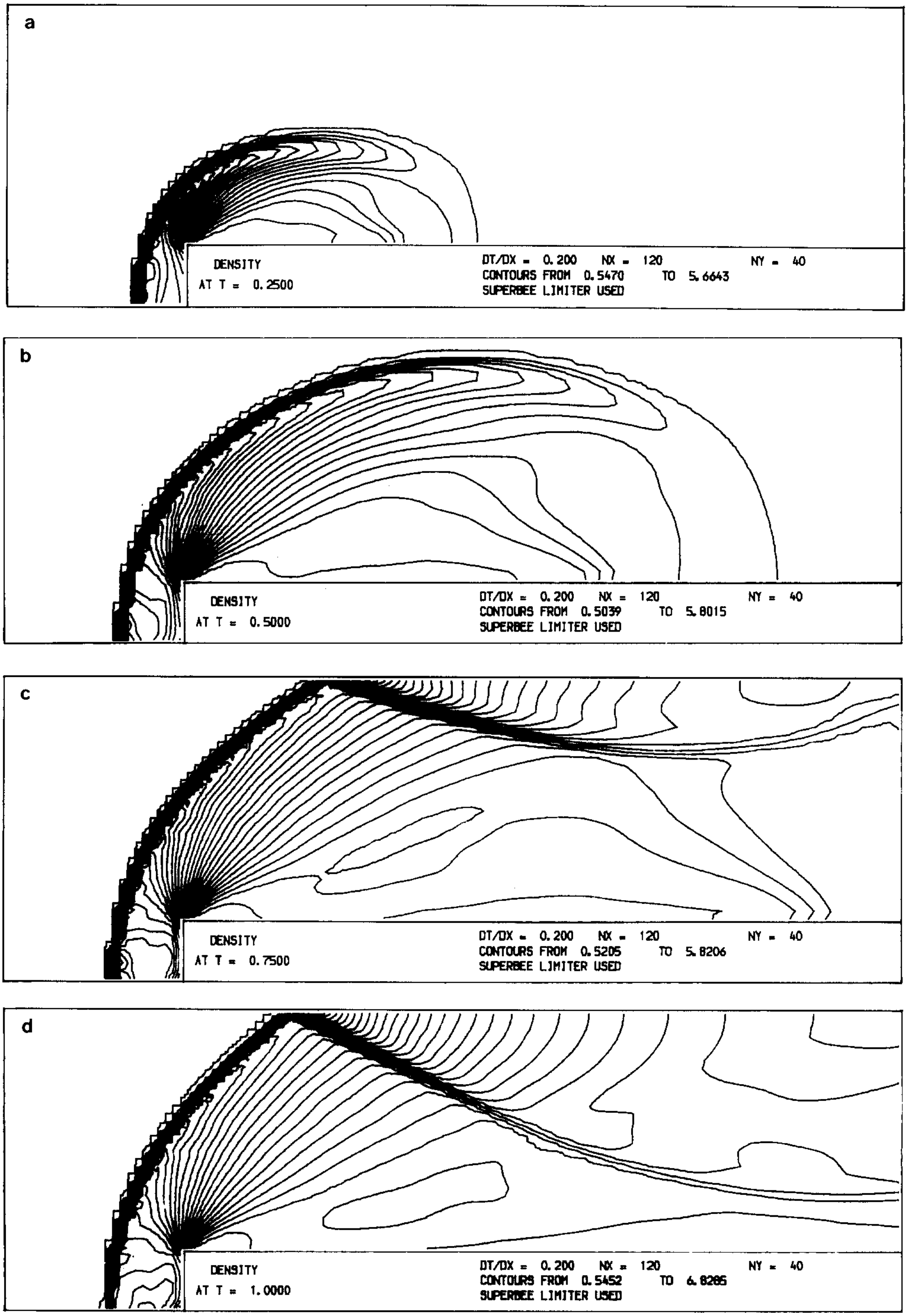

FIG. 2. Results for the ideal equation of state with $\gamma = \frac{7}{5}$: (a) at $t = 0.25$; (b) at $t = 0.5$; (c) at $t = 0.75$; (d) at $t = 1.0$; (e) at $t = 1.25$; (f) at $t = 1.5$; (g) at $t = 1.75$; (h) at $t = 2.0$.

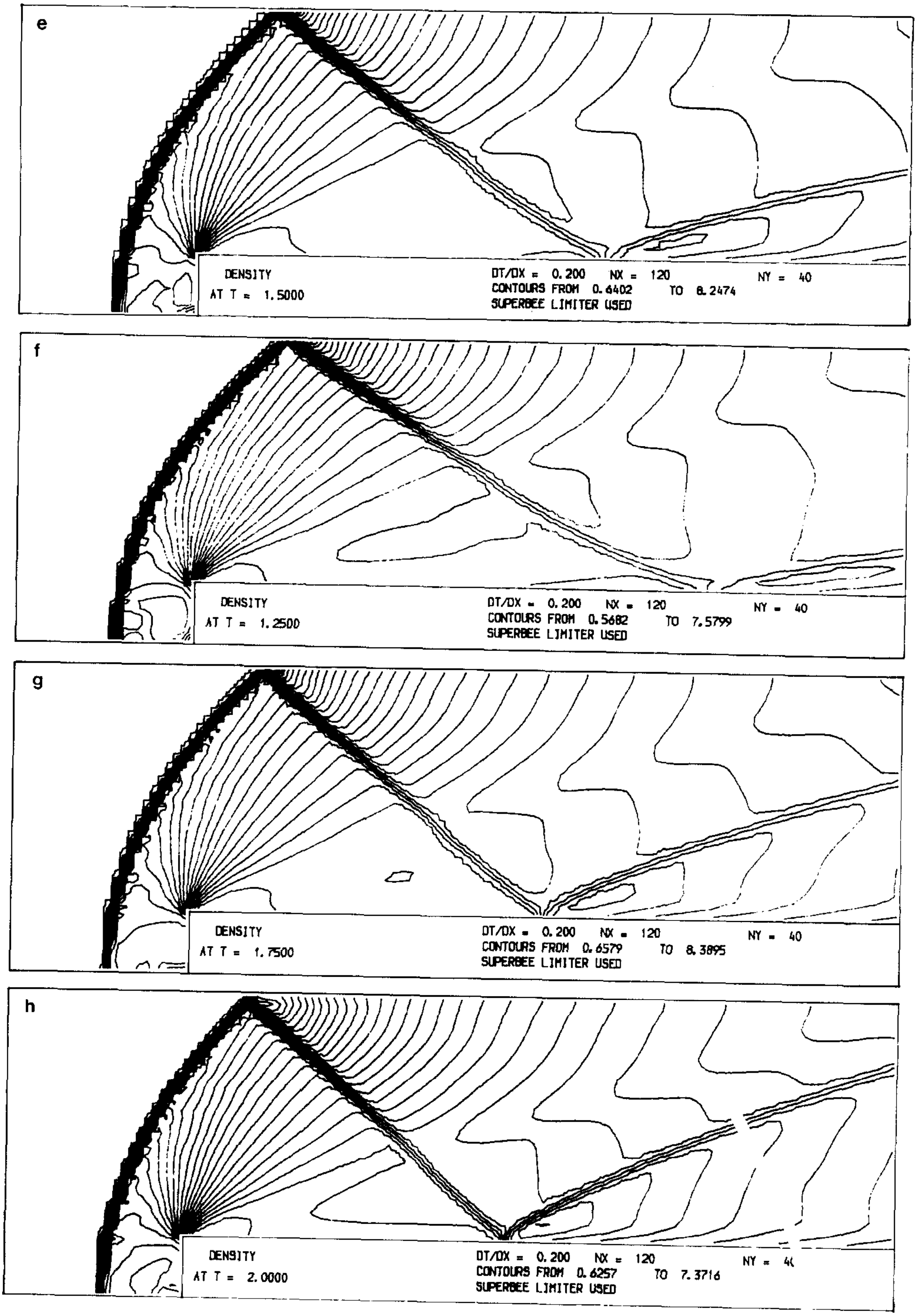

FIG. 2—*(continued)*.

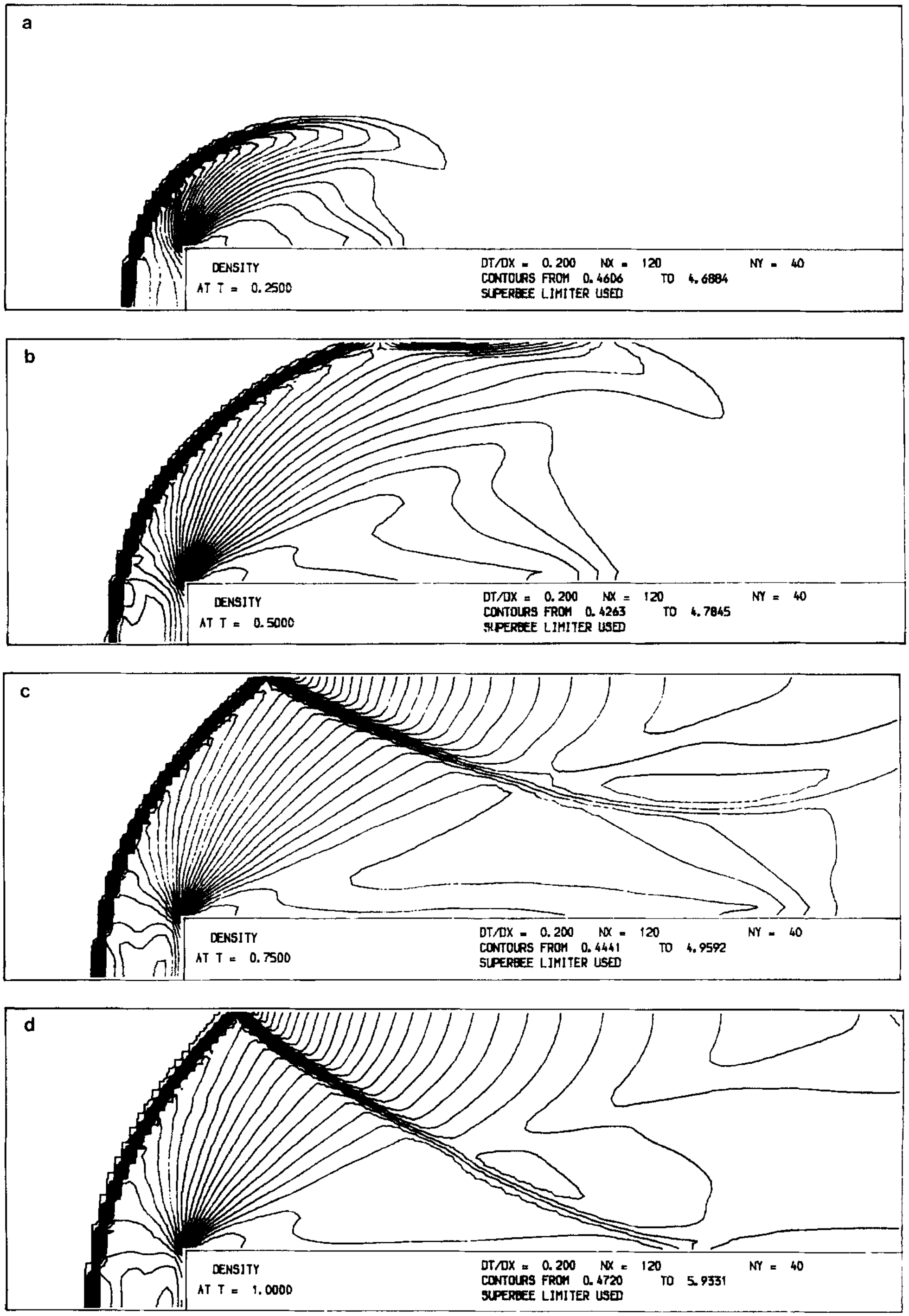

FIG. 3. Results for the stiffened equation of state with $\gamma = \frac{7}{5}$, $B = 1$: (a) at $t = 0.25$; (b) at $t = 0.5$; (c) at $t = 0.75$; (d) at $t = 1.0$; (e) at $t = 1.25$; (f) at $t = 1.5$; (g) at $t = 1.75$; (h) at $t = 2.0$.

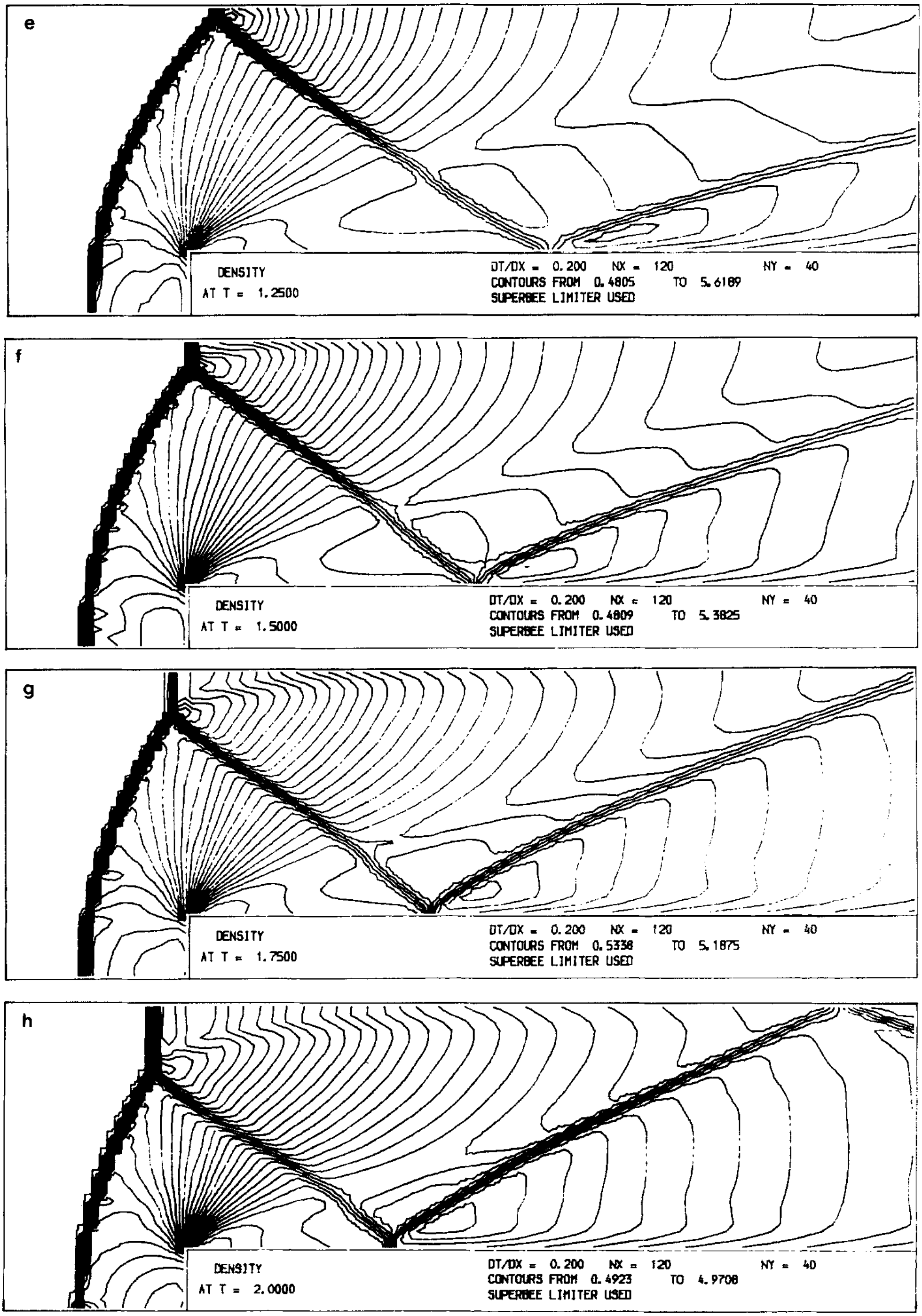

FIG. 3—*(continued)*.

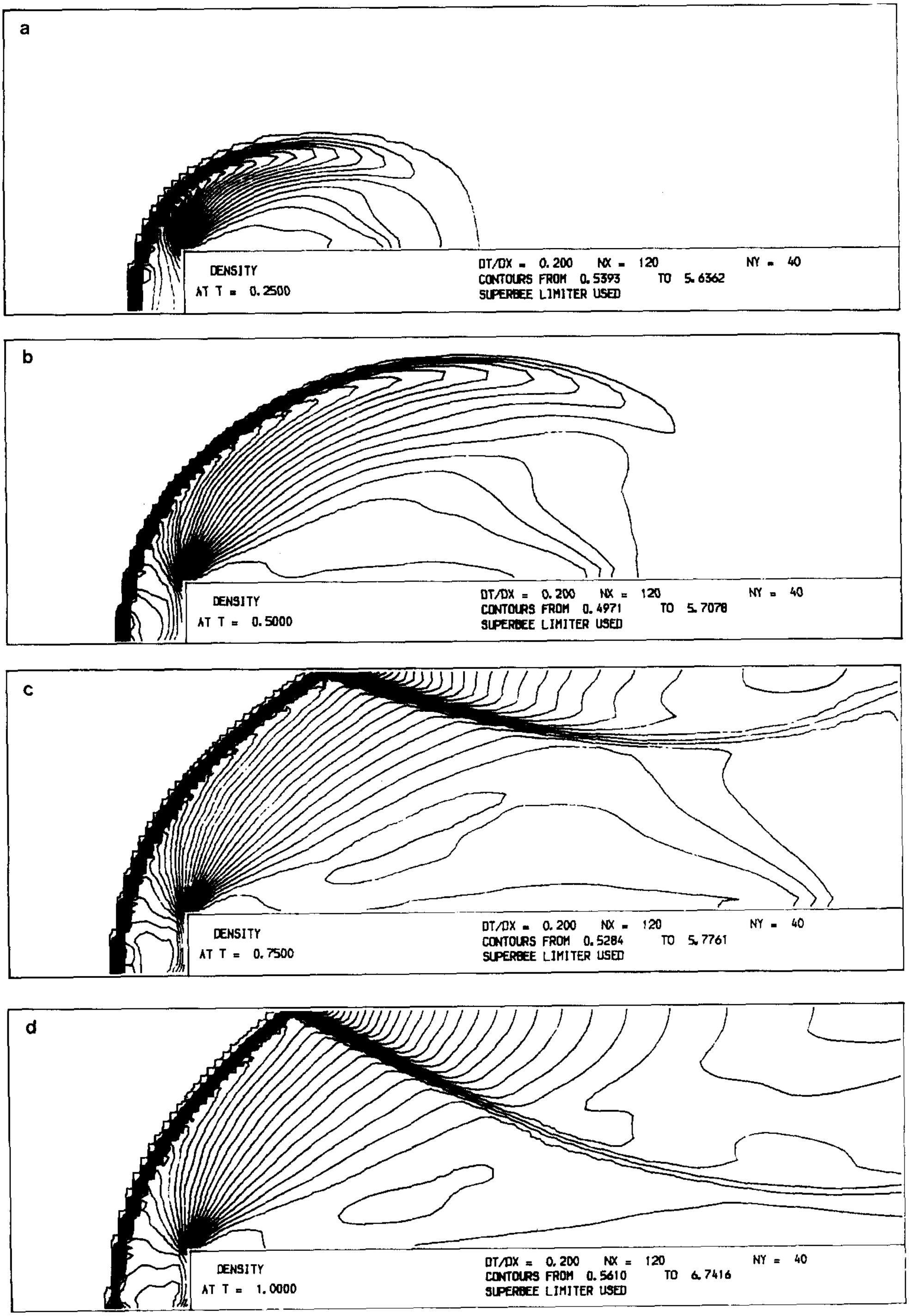

FIG. 4. Results for the "real air" equation of state at (a) $t = 0.25$; (b) $t = 0.5$; (c) $t = 0.75$; (d) $t = 1.0$; (e) $t = 1.25$; (f) $t = 1.5$; (g) $t = 1.75$; (h) $t = 2.0$.

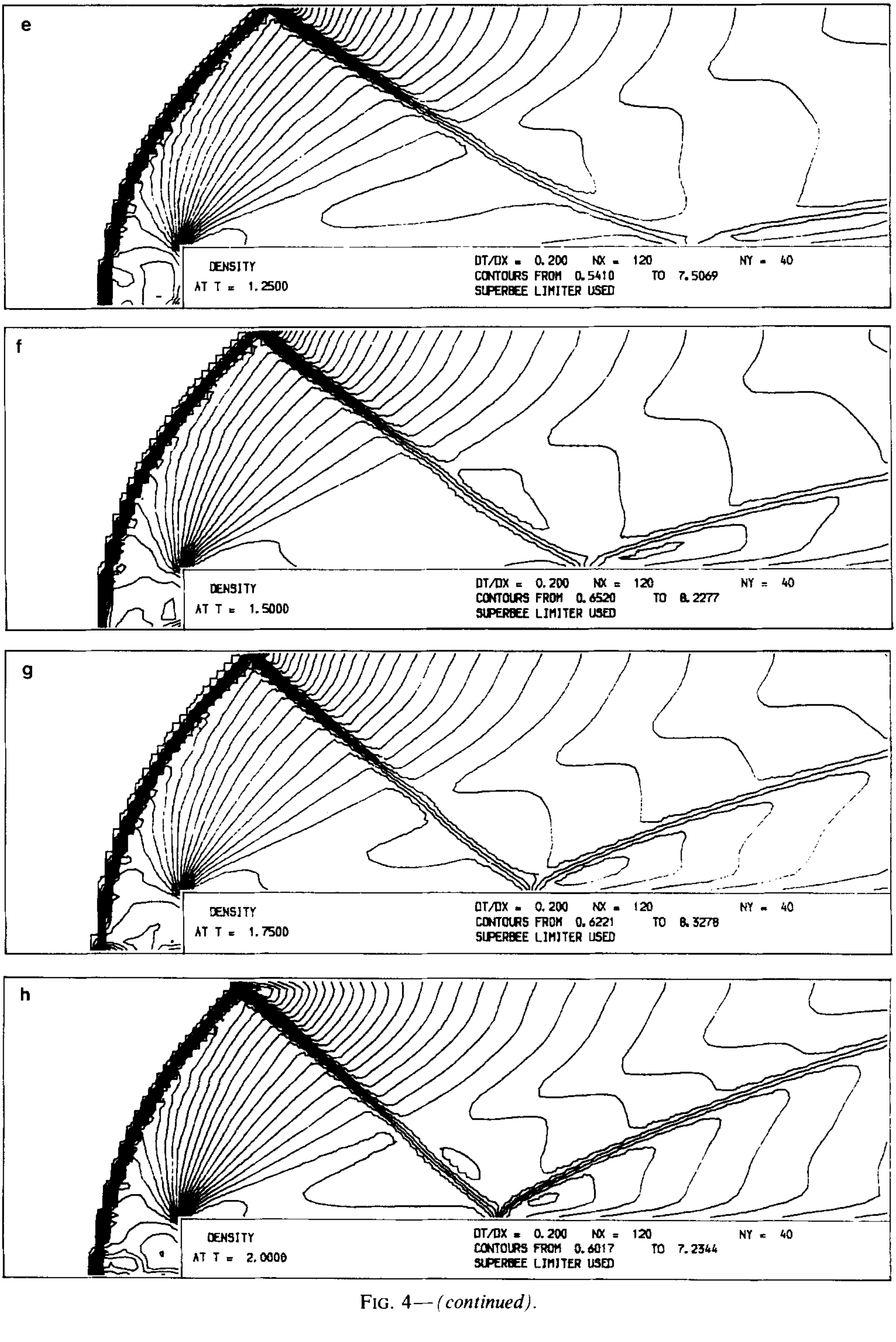

FIG. 4—*(continued)*.

together with

$$Y = \log_{10}(\rho/\rho_0)$$

$$Z = \log_{10}(i/i_0)$$

and ρ_0 is a reference density and i_0 is a reference internal energy. The constants a_i, $i = 1, ..., 24$, can be found in [6].

In case (a) we choose $\gamma = 1.4$ so that $i_0 = 25/14$, and in case (b) we choose $B = 1$, $\rho_0 = 1.4$, and $\gamma = 1.4$ so that $i_0 = \frac{25}{14}$. In case (c) we choose $\rho_0 = 1.4$ so that $i_0 = 5/7(a_1 - 1)$. The scheme employed is that of Section 3.

The first-order algorithm for updating the solution in the x-direction is that given in Section 3, with $\tilde{\lambda}_{1,2,3,4}$, $\tilde{\alpha}_{1,2,3,4}$, $\tilde{q}^2 = \tilde{u}^2 + \tilde{v}^2$, and $\tilde{\mathbf{e}}_{1,2,3,4}$ with the fourth component deleted. In addition, we can use the idea of flux limiters [7] to create a second-order algorithm which is oscillation free, and we can modify the scheme to disperse entropy violating solutions (see [8]). To advance the solution by a time Δt, we sweep through the mesh in the x, y, and x directions in turn, with time steps $\Delta t/4$, $\Delta t/2$, and $\Delta t/4$, respectively.

The main features of the solution are the Mach reflection of a bow shock at the upper wall, making the density distribution the most difficult to compute, and a rarefaction fan centred at the corner of the step.

Figures 2a–h refer to the ideal equation of state, Figs. 3a–h refer to the stiffened equation of state, and Figs. 4a–h refer to the equation of state for equilibrium air. In each case we take 120 mesh points in the x-direction and 40 mesh points in the y-direction, i.e., $\Delta x = \Delta y = \frac{1}{40}$. All computations have been done using a second-order entropy satisfying scheme with the "superbee" limiter (see [7]). The results for the density are output at times $t = j/4$, $j = 1, 2, ..., 8$, using a time step $\Delta t = 0.005$ so that the maximum C.F.L. number is 0.8. In each case 31 equally spaced contours have been drawn, i.e., at $\rho_{\min} + (i/30)(\rho_{\max} - \rho_{\min})$, $i = 0, 1, ..., 30$, where $\rho_{\max}$, $\rho_{\min}$ are the maximum and minimum densities throughout the flow, respectively.

Finally, we compare the c.p.u. time to compute the results obtained for the ideal gas case (a) using (i) Roe's original Riemann solver, and (ii) our general Riemann solver applied to the ideal gas case. (N.B. Although (i) and (ii) are solving the same problem, (ii) is for the general case and would therefore expect it to be more costly.) The comparison, using an Amdahl V7, is as follows:

(i) Using "superbee" with the modified entropy satisfying scheme and 120×40 mesh points takes 1.5 c.p.u. s to compute one time step and a total of 75 c.p.u. s to reach a real time of 0.25 s using 50 time steps.

(ii) Using "superbee" with the modified entropy satisfying scheme and 120×40 mesh points takes 1.65 c.p.u. s to compute one time step and a total of 82.5 c.p.u. s to reach a real time of 0.25 s using 50 time steps.

(N.B. For a 60×20 mesh the total c.p.u. time taken will be approximately $\frac{1}{8}$ of the values quoted above; e.g., in case (i) a total of 9.38 c.p.u. s would be required to reach a real time of 0.25 s.)

This show that our general Riemann solver, in conjunction with operator splitting, is only slightly more expensive than Roe's original.

4. CONCLUSIONS

We have extended the one-dimensional results of Glaister [2] to give a three-dimensional Riemann solver incorporating the technique of operator splitting. In doing so we have extended the scheme of Roe [3] in three dimensions to include a general equation of state. In addition, we have achieved satisfactory results for the problem of Mach 3 flow in a tunnel with a step and have seen that the algorithm is computationally efficient.

ACKNOWLEDGMENTS

I would like to express my thanks to Dr. M. J. Baines for useful discussions. I acknowledge the financial support of A.W.R.E. Aldermaston.

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